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Ordinary differential equations with superposition formulae: II. Parabolic subgroups of the symplectic group

A V Penskoi

Centre de recherches mathématiques, Université de Montréal, CP 6128, succ. Centre-ville, Montréal, Québec, H3C 3J7 Canada

E-mail: penskoi@crm.umontreal.ca

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Abstract

ODEs with superposition formulae related to parabolic subgroups of the complex symplectic group are explicitly found.

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Introduction

This paper is a continuation of a previous paper by the same author [1] as explained by the title of the present paper. We refer to the introduction and section 1 of [1] for preliminaries and necessary general information about ODEs with superposition formulae.

We consider the case when G is the complex symplectic group $Sp(2n)$ and M is a homogeneous space $Sp(2n)/P$, where P is an arbitrary parabolic subgroup of $Sp(2n)$. Thus, the action of G on G/P is (in general) non-primitive.

Some particular cases were investigated earlier by different authors. The case when P is a maximal parabolic subgroup was investigated in [2]. The choice of a maximal parabolic subgroup P corresponds to a primitive action of G. One particular choice of a maximal parabolic subgroup P gives a symplectic matrix Riccati equation investigated in detail in [3], where the superposition formulae are found.

Our goal is to construct explicitly the ODEs with superposition formulae for an arbitrary parabolic subgroup P. We will show a relation of these ODEs to the matrix Riccati equations.

The case of the symplectic groups turns out to be much more subtle than that of the $SL(N, \mathbb{C})$ groups. The geometry of the symplectic group is more complicated than that of the linear one. Already on the level of primitive group actions the corresponding nonlinear ODEs with superposition formulae will in general have quartic nonlinearities rather than quadratic ones. Thus we are no longer dealing only with matrix Riccati equations.

As a consequence the relatively simple matrix methods of [1] must be replaced by much more sophisticated ones. We shall make use of the methods of linear algebraic groups, i.e. the root systems, Bruhat paving, etc. The methods developed here for symplectic Lie groups are applicable to arbitrary complex simple Lie groups, be they symplectic, orthogonal, or exceptional.

1. The symplectic group, parabolic subgroups and generalized flag varieties

In this section we fix our conventions and recall some standard results about linear algebraic groups concerning the complex symplectic group $Sp(2n)$. Our basic sources are [4, 5].

By I_k we denote the identity $k \times k$ matrix. By J_k we denote a $k \times k$ matrix such that $(J_k)_{ij} = \delta_{i,k-i+1}$. We write simply I or J if we have no need to show the matrix size. By 0_k we denote the zero $k \times k$ matrix.

The symplectic group $Sp(2n)$ is a subgroup of $GL(2n)$ preserving a given antisymmetric non-degenerate bilinear form S. In this section we use the standard choice

$$S = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

but in the other sections we sometimes use other choices of S. Thus $Sp(2n) = \{X \in GL(2n) \mid XSX^T = S\}$.

As a maximal torus $T \subset Sp(2n)$, we choose a subgroup of symplectic matrices of the form $\operatorname{diag}\left(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\right)$. As a Borel subgroup $B \subset Sp(2n)$, we choose the subgroup of symplectic matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is an upper triangular matrix.

Define a basis of characters of T by

$$e_i\left(\operatorname{diag}\left(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\right)\right) = x_i.$$

Thus our root system R is

$$R = \{\pm 2e_i, \pm e_i \pm e_j \mid 1 \leqslant i, j \leqslant n, i \neq j\}$$

the system of positive roots R^+ is

$$R^+ = \{2e_i, e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\}$$

and the corresponding basis D is

$$D = \{e_i - e_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{2e_n\}.$$

Any parabolic subgroup is conjugate to a unique one containing B. Thus we consider only parabolic subgroups containing B.

Parabolic subgroups are in one-to-one correspondence with subsets $I \subset D$. The explicit description for the symplectic group is the following. Let us denote elements of D by numbers; $e_i - e_{i+1}$ corresponds to i and $2e_n$ corresponds to n. Thus we write $D = \{1, \ldots, n\}$. Consider $D - I$ and write the elements of $D - I$ as follows:

$$D - I = \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_s\}.$$

There are two cases:

Case I. The case $a_1 + \cdots + a_s = n$. Then the corresponding parabolic subgroup P consists of symplectic matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A_{ss} \end{pmatrix}$$

and A_{ij} is an $a_i \times a_j$ matrix.

Case II. The case $a_1 + \cdots + a_s \neq n$. Let $a_{s+1} = n - a_1 - \cdots - a_s$. Then the corresponding parabolic subgroup P consists of symplectic matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1,s+1} \\ 0 & A_{22} & \dots & A_{2,s+1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A_{s+1,s+1} \end{pmatrix}$$

A_{ij} is an $a_i \times a_j$ matrix,

$$C = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & 0 & 0 \\ 0 & \dots & 0 & C_{s+1,s+1} \end{pmatrix}$$

and C_{ij} is an $a_{s+1} \times a_{s+1}$ matrix.

By W we denote the Weyl group $W = N(T)/T$ and by $s_\alpha \in W$ we denote a reflection defined by $\alpha \in R$. By $\dot{w}$ we denote a representative of $w \in W = N(T)/T$ in $N(T) \subset Sp(2n)$. Let $l(w)$ be the length of $w \in W$.

Let $Y = Sp(2n)/P$, where P is a parabolic subgroup defined by $I \subset D$. Let

$$W_I' = \{w \in W \mid l(ws_\alpha) > l(w) \text{ for } \alpha \in I \subset D\}.$$

Now we can formulate the theorem which is most important for us.

Theorem 1. [4]

(i) *$Sp(2n)$ has the following decomposition*

$$Sp(2n) = \bigcup_{w \in W} B\dot{w}P.$$

(ii) *Let Y_w be the image of $B\dot{w}P$ under the canonical projection $Sp(2n) \to Y$. Then Y is a disjoint union of the Y_w, $w \in W_I'$.*

(iii) *$Y_w (w \in W_I')$ is a locally closed subvariety of Y isomorphic to $\mathbb{C}^{l(w)}$.*

Let w_1 be the longest element of W_I'. The explicit form of $\dot{w}_1$ is the following:

$$\dot{w}_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D depend on the case of parabolic subgroups mentioned above.

In case I $A = 0$, $D = 0$, C is a block-diagonal matrix $\operatorname{diag}\left(J_{a_1}, J_{a_2}, \ldots, J_{a_s}\right)$, and $B = -C$. In case II A and C are block-diagonal matrices, $A = \operatorname{diag}\left(0, \ldots, 0, I_{a_{s+1}}\right)$, $C = \operatorname{diag}\left(J_{a_1}, \ldots, J_{a_s}, 0_{a_{s+1}}\right)$, $D = A$ and $B = -C$.

Y_{w_1} is called the big Bruhat cell, $\dim Y_{w_1} = \dim Y$ and Y_{w_1} is isomorphic to $\mathbb{C}^{\dim Y}$. Y_{w_1} is the image of $B\dot{w}_1 P = R_u(P)\dot{w}_1 P$, where $R_u(P)$ is a unipotent radical of P isomorphic to $\mathbb{C}^{\dim Y}$. Thus each point p of Y_{w_1} is a P-coset in $B\dot{w}_1 P = R_u(P)\dot{w}_1 P$. In this coset there exists a unique element $A\dot{w}_1$, where $A \in R_u(P)$, or equivalently for each element $K \in B\dot{w}_1 P$ there exists a unique $L \in P$ such that $KL \in R_u(P)\dot{w}_1$.

2. Explicit formulae for the ODEs with superposition formulae

We have the canonical action of $Sp(2n)$ on the space $Y = Sp(2n)/P$ given by the formula

$$g \cdot hP = ghP \qquad g, h \in Sp(2n).$$

This action gives us a homomorphism from $\mathfrak{sp}(2n)$ to the algebra of vector fields on Y. Our goal is to find explicit formulae for this homomorphism using suitable coordinates on Y.

We can construct coordinates as follows: in the P-coset corresponding to $p \in Y_{w_1}$ we have a unique matrix of the form Aw_1, where $A \in R_u(P)$. Let us denote this matrix A by $K(p)$. If we have a fixed isomorphism $\Phi : R_u(P) \to \mathbb{C}^{\dim Y}$ then $\Phi(K(p))$ defines coordinates in $Y_{w_1} \subset Y$. Thus we have a class of coordinates in Y_{w_1} corresponding to isomorphisms between $R_u(P)$ and $\mathbb{C}^{\dim Y}$. Our natural intention is to find such a coordinate system that our homomorphism has a simple formula, so we must find a convenient isomorphism Φ. Let us concentrate now on $K(p)$, we will choose Φ later.

To describe the action of $Sp(2n)$ on Y we must find $K(gp)$. Let us consider a point $p \in Y_{w_1}$ and $g \in Sp(2n)$ such that $gp \in Y_{w_1}$. Let us take $K(p)w_1$ as a representative of p in the corresponding P-coset. After multiplication by g we obtain $gK(p)w_1$. We know that $K(gp)w_1$ is a unique element of the $R_u(P)w_1$ in the coset containing $gK(p)w_1$, thus there exists a unique element $L \in P$ such that

$$K(gp)w_1 = gK(p)w_1 L.$$

Consider a path $g(t) \in Sp(2n)$ such that $g(0) = I$, $\frac{\mathrm{d}}{\mathrm{d}t} g(t)|_{t=0} = X$ for some $X \in \mathfrak{sp}(2n)$. Let us find $\tilde{X} = \frac{\mathrm{d}}{\mathrm{d}t} K(g(t)p)|_{t=0}$.

$$\begin{aligned} \tilde{X} &= \frac{\mathrm{d}}{\mathrm{d}t} g(t)K(p)w_1 L(t) w_1^{-1}\Big|_{t=0} \\ &= g'(t)K(p)w_1 L(t)w_1^{-1}\Big|_{t=0} + g(t)K(p)w_1 L'(t)w_1^{-1}\Big|_{t=0} \\ &= XK(p) + K(p)w_1 L' w_1^{-1} \end{aligned}$$

where $L' = L'(0) \in \mathfrak{p}$ is uniquely determined by the condition that $\tilde{X} \in \mathfrak{u}$, where by $\mathfrak{u}$ we denote the Lie algebra of $R_u(P)$.

Thus if we want to find $\tilde{X}$ given $K(p)$ and X, we have to find L'. Let us do a change of basis. The key idea of this change is transforming $R_u(P)$ to some subgroup of lower-triangular matrices. Let us do the change of basis defined by

$$F = \begin{pmatrix} 0 & I \\ R & 0 \end{pmatrix}$$

where R is a block-antidiagonal matrix

$$R = \begin{pmatrix} 0 & \dots & I_{a_s} \\ \vdots & ⋰ & \vdots \\ I_{a_1} & \dots & 0 \end{pmatrix} \qquad \text{or} \qquad R = \begin{pmatrix} 0 & \dots & I_{a_{s+1}} \\ \vdots & ⋰ & \vdots \\ I_{a_1} & \dots & 0 \end{pmatrix}$$

in cases I and II, respectively.

If $A \in Sp(2n)$ preserves S, i.e. $ASA^T = S$, then $A_1 = FAF^{-1}$ preserves $S_1 = FSF^T$, i.e. $A_1 S_1 A_1^T = S_1$. Thus A_1 is an element of the group $Sp(2n, S_1)$ of matrices preserving S_1.

Using our transformation we obtain

$$\tilde{X}_1 = X_1 K_1(p) + K_1(p) L_1' \tag{1}$$

where $X_1 = FXF^{-1}$, $K_1(p) = FK(p)F^{-1}$, $L_1' = Fw_1 L' w_1^{-1} F^{-1}$ and $\tilde{X}_1 = F\tilde{X}F^{-1}$.

Thus X_1 is an element of $\mathfrak{sp}(2n, S_1)$, i.e. the Lie algebra of $Sp(2n, S_1)$, $\mathfrak{sp}(2n, S_1) = \{X \mid XS_1 + S_1 X^T = 0\}$.

The matrix $K_1(p)$ is an element of $\tilde{K} = FR_u(P)F^{-1}$. It is easy to check that $\tilde{K}$ is a subgroup of $Sp\,(2n, S_1)$ consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where in case I, A and C are block lower-diagonal matrices,

$$A = \begin{pmatrix} I_{a_1} & & 0 \\ & \ddots & \\ * & & I_{a_s} \end{pmatrix} \qquad C = \begin{pmatrix} I_{a_s} & & 0 \\ & \ddots & \\ * & & I_{a_1} \end{pmatrix}.$$

In case II A and C are also block lower-diagonal matrices,

$$A = \begin{pmatrix} I_{a_1} & & 0 \\ & \ddots & \\ * & & I_{a_{s+1}} \end{pmatrix} \qquad C = \begin{pmatrix} I_{a_{s+1}} & & 0 \\ & \ddots & \\ * & & I_{a_1} \end{pmatrix}$$

but in case II there is also a condition on B. B must be of the form

$$\begin{pmatrix} * & 0_{a_{s+1}} \\ * & * \end{pmatrix}.$$

The matrix L_1' is an element of $\tilde{P} = Fw_1 P w_1^{-1} F^{-1}$. It is easy to check that $\tilde{P}$ is a subgroup of $Sp\,(2n, S_1)$ consisting of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A and D are block upper-diagonal matrices,

$$A = \begin{pmatrix} A_{11} & \dots & A_{1q} \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{qq} \end{pmatrix} \qquad D = \begin{pmatrix} D_{11} & \dots & D_{1q} \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_{qq} \end{pmatrix},$$

$q = s$ in case I and $q = s + 1$ in case II, A_{ij} is a $a_i \times a_j$ matrix, D_{ij} is a $a_{s-i+1} \times a_{s-j+1}$ matrix in case I or a $a_{s-i+2} \times a_{s-j+2}$ matrix in case II, in case I C is a zero matrix, in case II C has the form

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

where * is some $a_{s+1} \times a_{s+1}$ matrix.

The matrix $\tilde{X}_1$ is an element of the Lie algebra of $\tilde{K}$, we will denote this algebra by $\tilde{\mathfrak{k}}$.

Now our problem is the following: given $X_1 \in \mathfrak{sp}(2n, S_1)$ and $K_1(p) \in \tilde{K}$, find $\tilde{X}_1 \in \tilde{\mathfrak{k}}$ using formula (1), where L_1' is some element of $\tilde{P}$ uniquely determined by the condition $\tilde{X}_1 \in \tilde{\mathfrak{k}}$.

With this we are now able to find L_1' explicitly. Let us divide X_1 into blocks

$$X_1 = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix}$$

where the blocks X_{ij} are defined in the following way: in case I we have $r = 2s$, X_{ij} is a $b_i \times b_j$ matrix, where $b_i = a_i$ for $1 \leqslant i \leqslant s$ and $b_i = a_{2s-i+1}$ for $s+1 \leqslant i \leqslant 2s$, in case II we have $r = 2s+2$, X_{ij} is a $b_i \times b_j$ matrix, where $b_i = a_i$ for $1 \leqslant i \leqslant s+1$ and $b_i = a_{2s-i+3}$ for $s+2 \leqslant i \leqslant 2s+2$.

Let us divide $K_1(p)$ into blocks in the same way, then we obtain

$$K_1(p) = \begin{pmatrix} I_{b_1} & 0 & \dots & 0 \\ K_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ K_{r,1} & \dots & K_{r,r-1} & I_{b_r} \end{pmatrix}$$

where r and b_i are defined as above, K_{ij} is a $b_i \times b_j$ matrix, and in case II it is necessary to replace $K_{s+2,s+1}$ by the zero matrix of the same size.

Let us also divide L_1' into blocks in the same way, we obtain

$$L_1' = \begin{pmatrix} L_{11} & \dots & \dots & L_{1,2s} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & L_{2s,2s} \end{pmatrix}$$

in case I, and

$$L_1' = \begin{pmatrix} L_{11} & \dots & \dots & \dots & \dots & \dots & \dots & L_{1,2s} \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & & & \vdots \\ \vdots & & 0 & L_{s+1,s+1} & & & & \vdots \\ \vdots & & 0 & L_{s+2,s+1} & L_{s+2,s+2} & & & \vdots \\ \vdots & & 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & L_{2s,2s} \end{pmatrix}$$

in case II. In both cases L_{ij} is a $b_i \times b_j$ matrix, b_i is defined above.

In the following lemma and theorem we use the convention that for a matrix A one has $A^0 = I$.

Lemma 1. *L_1' can be found using the following formulae.*

In case II *for* $i \neq s+1$ *and in case* I *we have*

$$\begin{pmatrix} L_{1,i} \\ \vdots \\ L_{i,i} \end{pmatrix} = -\left(\sum_{k=0}^{i-1}(-1)^k M^k\right)\left[\begin{pmatrix} X_{1,i} \\ \vdots \\ X_{i,i} \end{pmatrix} + \begin{pmatrix} X_{1,i+1} & \dots & X_{1,r} \\ \vdots & & \vdots \\ X_{i,i+1} & \dots & X_{i,r} \end{pmatrix}\begin{pmatrix} K_{i+1,i} \\ \vdots \\ K_{r,i} \end{pmatrix}\right]$$

where

$$M = \begin{pmatrix} 0_{b_1} & \dots & \dots & 0 \\ K_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ K_{i,1} & \dots & K_{i,i-1} & 0_{b_i} \end{pmatrix}$$

b_i *and* r *are defined above, and in case* II *if* $K_{s+2,s+1}$ *is in* M*, then replace* $K_{s+2,s+1}$ *by the zero matrix of the same size.*

In case II *when* $i = s+1$ *we have*

$$\begin{pmatrix} L_{1,s+1} \\ \vdots \\ L_{s+2,s+1} \end{pmatrix} = -\left(\sum_{k=0}^{s+1}(-1)^k M^k\right)\left[\begin{pmatrix} X_{1,s+1} \\ \vdots \\ X_{s+2,s+1} \end{pmatrix} + \begin{pmatrix} X_{1,s+3} & \dots & X_{1,2s+2} \\ \vdots & & \vdots \\ X_{s+2,s+3} & \dots & X_{s+2,2s+2} \end{pmatrix}\begin{pmatrix} K_{s+3,s+1} \\ \vdots \\ K_{2s+2,s+1} \end{pmatrix}\right]$$

where

$$M = \begin{pmatrix} 0_{b_1} & 0 & \dots & \dots & 0 \\ K_{21} & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ K_{s+1,1} & \dots & K_{s+1,s-1} & 0_{b_{s+1}} & \vdots \\ K_{s+2,1} & \dots & K_{s+2,s} & 0 & 0_{b_{s+2}} \end{pmatrix}. \tag{2}$$

Proof. Let us consider case I, case II is analogous.

Let us divide $\tilde{X}_1$ in blocks $\tilde{X}_{ij}$ in the same way as X_1. Let us consider an ith column of blocks of $\tilde{X}_1$. From (1) we have

$$\begin{pmatrix} \tilde{X}_{1,i} \\ \vdots \\ \tilde{X}_{2s,i} \end{pmatrix} = X_1 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{b_i} \\ K_{i+1,i} \\ \vdots \\ K_{2s,i} \end{pmatrix} + K_1(p)\begin{pmatrix} L_{1,i} \\ \vdots \\ L_{i,i} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3}$$

It follows from the condition $\tilde{X}_1 \in \tilde{\mathfrak{k}}$ that $\tilde{X}_{1,i} = 0, \dots, \tilde{X}_{i,i} = 0$. Using (3) we obtain from these conditions that

$$0 = \begin{pmatrix} X_{1,i} \\ \vdots \\ X_{i,i} \end{pmatrix} + \begin{pmatrix} X_{1,i+1} & \dots & X_{1,2s} \\ \vdots & & \vdots \\ X_{i,i+1} & \dots & X_{i,2s} \end{pmatrix}\begin{pmatrix} K_{i+1,i} \\ \vdots \\ K_{2s,i} \end{pmatrix} + \begin{pmatrix} I_{b_1} & \dots & \dots & 0 \\ K_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ K_{i,1} & \dots & K_{i,i-1} & I_{b_i} \end{pmatrix}\begin{pmatrix} L_{1,i} \\ \vdots \\ L_{i,i} \end{pmatrix}.$$

Using the fact that $(I+A)^{-1} = \sum_{k=0}^{\infty}(-1)^k A^k$ if A is a nilpotent matrix, we can solve this linear equation and obtain the lemma statement. □

Now we are able to find $\tilde{X}_1$ explicitly in terms of blocks of X_1 and $K_1(p)$. So we see that it is more convenient to choose our isomorphism Φ using $K_1(p)$. One can check that as coordinates we can take the blocks K_{ij} for $2 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant \min(i-1, r-i)$ and the 'symmetric parts', $K_{i,2s-i+1} + K^T_{i,2s-i+1}$ for $s+1 \leqslant i \leqslant 2s$ in case I, and $K_{i,2s-i+3} + K^T_{i,2s-i+3}$ for $s+3 \leqslant i \leqslant 2s+2$ in case II. However, it is more convenient to write our ODEs using all the matrices K_{ij} as variables but with constraints given by the condition $K_1(p) \in Sp(2n, S_1)$.

Thus, using (1), formulae from the lemma 1, and the condition $K_1(p) \in Sp(2n, S_1)$ we obtain the final formulae for our ODEs.

Theorem 2. *The equations with superposition formulae related to the canonical action of $Sp(2n)$ on $Sp(2n)/P$ are the following.*

In case I *we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}K_{i+1,i}\\ \vdots\\ K_{r,i}\end{pmatrix} &= \begin{pmatrix}X_{i+1,i}\\ \vdots\\ X_{r,i}\end{pmatrix} + \begin{pmatrix}X_{i+1,i+1} & \dots & X_{i+1,r}\\ \vdots & & \vdots\\ X_{r,i+1} & \dots & X_{r,r}\end{pmatrix}\begin{pmatrix}K_{i+1,i}\\ \vdots\\ K_{r,i}\end{pmatrix}\\ &\quad - \begin{pmatrix}K_{i+1,1} & \dots & K_{i+1,i}\\ \vdots & & \vdots\\ K_{r,1} & \dots & K_{r,i}\end{pmatrix}\left[\sum_{k=0}^{i-1}(-1)^k\begin{pmatrix}0_{b_1} & 0 & \dots & 0\\ K_{21} & \ddots & \ddots & \vdots\\ \vdots & & \ddots & 0\\ K_{i1} & \dots & K_{i,i-1} & 0_{b_i}\end{pmatrix}^k\right]\\ &\quad \times\left[\begin{pmatrix}X_{1,i}\\ \vdots\\ X_{i,i}\end{pmatrix} + \begin{pmatrix}X_{1,i+1} & \dots & X_{1,r}\\ \vdots & & \vdots\\ X_{i,i+1} & \dots & X_{i,r}\end{pmatrix}\begin{pmatrix}K_{i+1,i}\\ \vdots\\ K_{r,i}\end{pmatrix}\right]\end{aligned} \tag{4}$$

where $1 \leqslant i \leqslant 2s-1$, r and b_i are defined above, X_{ij} are functions of t such that $X_1 = (X_{ij}) \in \mathfrak{sp}(2n, S_1)$.

In case II *the equation is given for $1 \leqslant i \leqslant 2s+1$, $i \neq s+1$ by the same formulae* (4)*, but $K_{s+2,s+1}$ must be replaced by a zero matrix of the same size. In the case $i = s+1$ we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}K_{s+3,s+1}\\ \vdots\\ K_{2s+2,s+1}\end{pmatrix} &= \begin{pmatrix}X_{s+3,s+1}\\ \vdots\\ X_{2s+2,s+1}\end{pmatrix} + \begin{pmatrix}X_{s+3,s+3} & \dots & X_{s+3,2s+2}\\ \vdots & & \vdots\\ X_{2s+2,s+3} & \dots & X_{2s+2,2s+2}\end{pmatrix}\begin{pmatrix}K_{s+3,s+1}\\ \vdots\\ K_{2s+2,s+1}\end{pmatrix}\\ &\quad - \begin{pmatrix}K_{s+3,1} & \dots & K_{s+3,s+2}\\ \vdots & & \vdots\\ K_{2s+2,1} & \dots & K_{2s+2,s+2}\end{pmatrix}\left[\sum_{k=0}^{s+1}(-1)^k M^k\right]\\ &\quad \times\left[\begin{pmatrix}X_{1,s+1}\\ \vdots\\ X_{s+2,s+1}\end{pmatrix} + \begin{pmatrix}X_{1,s+3} & \dots & X_{1,2s+2}\\ \vdots & & \vdots\\ X_{s+2,s+3} & \dots & X_{s+2,2s+2}\end{pmatrix}\begin{pmatrix}K_{s+3,s+1}\\ \vdots\\ K_{2s+2,s+1}\end{pmatrix}\right]\end{aligned}$$

where M is the same matrix as in formula (2).

The variables K_{ij} are not independent, they satisfy the quadratic constraint $K_1(p)S_1K_1^T(p) = S_1$.

Proof. Substitute the results of lemma 1 in formula (1). $\square$

Now let us remark that in case I our ODEs given by the formulae (4) coincide with the formulae for the ODEs with superposition formulae related to the canonical action of $SL(2n)$ on a homogeneous space $SL(2n)/p$ (see [1]), but with the quadratic constraints added. So we can remark as in [1] that the system of equations (4) can be reduced to the matrix Riccati equations. Let us formulate this result exactly. Let us recall that a matrix Riccati equation is an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = A(t) + B(t)W(t) + W(t)C(t) + W(t)D(t)W(t) \tag{5}$$

where A, B, C, D and W are matrices of appropriate sizes. The matrix Riccati equation is an equation with a superposition formula. Let us define matrices W_i as

$$W_i = \begin{pmatrix} K_{i+1,i} \\ \cdots \\ K_{r,i} \end{pmatrix}.$$

Then we have the following proposition.

Proposition 1. *The equations* (4), $1 \leqslant i \leqslant 2s - 1$, *giving the ODEs with superposition formulae in case* I, *can be rewritten in the form*

$$\frac{\mathrm{d}}{\mathrm{d}t}W_i = A_i + B_iW_i + W_iC_i + W_iD_iW_i$$

where $A_1, \ldots, D_1$ *are functions of* t, *and for* $i > 1$ $A_i, \ldots, D_i$ *are functions of* t *and* $W_1, \ldots, W_{i-1}$. *Thus we obtain the system of matrix Riccati equations* (5) *for* W_i *such that in the equation for* W_i *the matrices* $A_i, \ldots, D_i$ *depend on the 'previous' matrices* $W_1, \ldots, W_{i-1}$. *There are also quadratic constraints on* W_i.

The proof of this proposition and the explicit form of $A_i, \ldots, D_i$ can be found in [1]. It is necessary to remark that the coefficients $A_i, \ldots, D_i$ are not arbitrary, there are conditions for them, see our example below.

Thus in case I the solving of our ODE can be reduced to the solving of matrix Riccati equations.

Unfortunately, it is not so simple in case II because proposition 1 is not true. Indeed, for $i \neq s+1$ we have the same thing but it is impossible to write the equation for W_{s+1} in the form described in proposition 1.

The simplest example is the parabolic subgroup corresponding to $s = 1$, $a_1 = n$. This corresponds to case I in our classification. We have

$$K_1 = \begin{pmatrix} I_n & 0 \\ K_{21} & I_n \end{pmatrix}.$$

Equation (4) in this case is the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}K_{21} = X_{21} + X_{22}K_{21} - K_{21}X_{11} - K_{21}X_{12}K_{21}. \tag{6}$$

The constraint $K_1(p)S_1K_1^T(p) = S_1$ in this case is not quadratic (as in general), it is linear, namely $K_{21}^T - K_{21} = 0$, i.e. K_{21} is symmetric. The condition $X_1 \in \mathfrak{sp}(2n, S_1)$ means that X_{12} and X_{21} are symmetric and $X_{22} = -X_{11}^T$. Thus we see that this case gives us the symplectic Riccati equation studied in [3]:

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = A(t) + B(t)W(t) + W(t)C(t) + W(t)D(t)W(t)$$

where W, A and D are symmetric, and $B^T = C$. Indeed, it is sufficient to substitute in (6) $K_{21} = W$, $X_{21} = A$, $X_{22} = B$, $X_{11} = -C$ and $X_{12} = -D$. The conditions that W, A and D are symmetric and $B^T = C$ follow from the conditions on K_{21} and X_{ij}.

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